MATH 732: CUBIC HYPERSURFACES

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1. PROBLEMS ON CHERN CLASSES

Here is a series of exercises introducing the foundations of Chern classes. We remark that there are many ways to define Chern classes, but here we follow the approach of Grothendieck in [Gro58] (there are translations available online). Here we first give the axioms and then outline the construction through exercises. The best problems probably involve applications of the splitting principle, so one strategy could be to work on those problems first after reading the worksheet.

Setup. Suppose $X \mapsto A^*(X)$ is a contravariant functor from the category of smooth projective k-varieties to *graded commutative rings* (that have a 1) and assume there is a natural transformation sending

$$\mathrm{cl} \colon \mathrm{Pic}(X) \to A^2(X).$$

Lastly, for any smooth closed subvariety $i \colon Y \hookrightarrow X$ of codimension r assume there is a group homomorphism:

$$i_* \colon A^*(Y) \to A^*(X)$$

that raises the degree by $2r$. Define:

$$[Y]_A = i_*(1_Y) \in A^{2r}(X).$$

Remark 1.1. There are a few great models to have in mind.

(1) $A^{2*} = \mathrm{CH}^*(-)$,
(2) $A^* = \mathrm{H}^*(-, \mathbf{Z})$, or
(3) $A^*(X) = \bigoplus_{p+q=*} \mathrm{H}^q(X, \wedge^p \Omega_X)$.

Given such a functor, natural transformation, and plethora of group homomorphisms, we FURTHER assume the following axioms hold:

A. (Projective bundle formula) For a rank p vector bundle $\mathcal{E}$ on a smooth projective variety X with projective bundle $\mathbf{P}(E)$. Let $\xi_{\mathcal{E}} \in A^2(\mathbf{P}(\mathcal{E}))$ be the image of the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Assume that the elements

$$(\xi_{\mathcal{E}})^0, \cdots, (\xi_{\mathcal{E}})^{p-1}$$

form a free $A^*(X)$-basis for $A^*(\mathbf{P}(\mathcal{E}))$.

B. (Fundamental class) If $\mathcal{L}$ is a line bundle on X and $s \in \mathrm{H}^0(X, \mathcal{L})$ is a section that is transverse to the 0-section and $i\colon Y \subseteq X$ is the corresponding divisor then

$$[Y]_A = \mathrm{cl}(L) \in A^2(X)$$

SLOGAN: a line bundle is represented by the fundamental class of its divisor.

C. (Functoriality of inclusion morphisms) For a series of inclusions $i\colon Z \hookrightarrow Y$ and $j\colon Y \hookrightarrow X$:

$$(j \circ i)_* = j_* \circ i_*.$$

D. (Projection formula) For an inclusion $i\colon Y \hookrightarrow X$, any class $a \in A^*(X)$ and any class $b \in A^*(X)$:

$$i_*(bi^*(a)) = i_*(b)a.$$

Exercise 1. *Prove that if $\mathcal{E}$ is a rank 2 vector bundle on X with a section s that is transverse to the 0-section of $\mathcal{E}$. Let*

$$i\colon Y = (s = 0) \hookrightarrow X.$$

Assume $\mathcal{E}$ is an extension of line-bundles:

$$0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0.$$

Define:

$$\xi_i = \mathrm{cl}(\mathcal{L}_i) \in A^2(X).$$

Assume (for convenience of proof) that the induced section of $\mathcal{L}_2$ is transverse to the 0-section. Prove that

$$[Y]_A = \xi_1 \cdot \xi_2 \in A^4(X).$$

(Similarly, one can prove a result for a filtered vector bundle of higher rank.) What does this say if the section is nowhere vanishing?

Exercise 2. *Given rank r vector bundle $\mathcal{E}$ on X, prove that there is a smooth projective variety $\mu\colon Y \to X$ such that*

(1) $\mu^*\mathcal{E}$ is a successive extension of line bundles, and
(2) $\mu^*\colon A^*(X) \to A^*(Y)$ is injective.

Exercise 3 (Definition of Chern classes in A). *Let $\mathcal{E}$ be a rank p vector bundle on X. Prove that there are unique elements:*

$$c_i(\mathcal{E}) \in A^{2i}(X)$$

such that

$$\sum_{i=0}^{p} c_i(\mathcal{E})(\xi_\mathcal{E})^{p-i} = 0 \in A(\mathbf{P}(\mathcal{E})).$$

(where $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ for $i > p$). The element:

$$c_\bullet(X) = \sum c_i(\mathcal{E}) \in A^*(X)$$

is called the total Chern class of $\mathcal{E}$. Prove that if $\mathcal{L}$ is a rank 1 vector bundle on X then $c_\bullet(\mathcal{L}) = 1 + \mathrm{cl}(\mathcal{L})$.

Remark 1.2. So, Grothendieck's perspective is that Chern classes are defined via the multiplicative structure on projective bundles. Grothendieck further shows that the Chern classes are functorial, and if

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is a short exact sequence of vector bundles then:

$$c_\bullet(\mathcal{E}_2) = c_\bullet(\mathcal{E}_1) c_\bullet(\mathcal{E}_3). \qquad \text{(Additivity)}$$

Remark 1.3. The *splitting principle* says that to compute Chern class formulas we may assume that our vector bundle is an extension of line bundles (or even a direct sum of line bundles) and then try to compute the Chern classes in terms of this splitting. The logic behind the formula is that as in Exercise 2, we may pull back the vector bundle to a variety where it is an extension of line bundles, and any formula follows from the above additivity principle.

Exercise 4. *Let X be a variety and $\mathcal{E}$ a vector bundle of rank r. Use the splitting principle to compute the following Chern classes.*

(1) $c_\bullet(\mathcal{E}^\vee) = 1 - c_1(\mathcal{E}) + c_2(\mathcal{E}) - \cdots$,
(2) $c_\bullet(\mathrm{Sym}^2(\mathcal{E}))$,
(3) $c_\bullet(\mathcal{E}^{\otimes 2})$,
(4) $c_\bullet(\wedge^2 \mathcal{E})$,
(5) $c_\bullet(\wedge^r \mathcal{E})$,
(6) $c_\bullet(\mathcal{E} \otimes \mathcal{L})$ for a line bundle $\mathcal{L}$.

Exercise 5. *Compute the total Chern classes of $T_{\mathbf{P}^n}$ and $T_{\mathbf{P}^1 \times \mathbf{P}^1}$.*

REFERENCES

[Gro58] Alexander Grothendieck. The class theory of Chern. *Bulletin de la Société Mathématique de France*, 86:137–154, 1958.